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Incorporating inventory control decisions into a strategic distribution network design model with stochastic demand

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Abstract

In this paper, we propose a simultaneous approach to incorporate inventory control decisions—such as economic order quantity and safety stock decisions—into typical facility location models, which are used to solve the distribution network design problem. A simultaneous model is developed considering a stochastic demand, modeling also the risk pooling phenomenon. We present a non-linear-mixed-integer model and a heuristic solution approach, based on Lagrangian relaxation and the sub-gradient method. In a numerical application, we found that the potential cost reduction, compared to the traditional approach, increases when the holding costs and/or the variability of demand are higher.

Keywords: Supply chain management; Distribution network design; Facility location problems; Inventory control; Risk pooling; Lagrangian relaxation

1. Introduction

The standard literature on supply chain management classifies the problems into three hierarchical levels: strategic (long term), tactical (medium term), and operational (short term), though the limits between each level remain unclear. The usual approach to solve these problems has typically been to tackle them in isolation from one another. In practice, strategic decisions are made by top managers, while the tactical and operational decisions are made by bottom level

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managers. This situation tends to promote incompatibilities and incoherence between each level. For instance, facility location problems are considered as strategic, imposing a strong simplification regarding tactical and operational aspects directly related to the optimal location. Examples of these tactical/operational aspects are the inventory control policy, the choice of transportation mode/capacity, warehouse design and management, vehicle routing, among others.

The aim of this paper is to incorporate tactical/operational decisions into to the facility location problem solution scheme. Specifically, inventory management decisions will be simultaneously modeled with the distribution network design. This inclusion acquires especial relevance in the presence of high holding costs (e.g. frozen food industry) and high-variability demands. An example is provided to illustrate the relevance of this issue. Fig. 1 shows a distribution network where a single plant supplies products to regional warehouses, and these distribute products to retailers or customers. The ownership of the chain is assumed to belong to a single decision maker responsible for the holding cost at each facility, as well as all the transportation costs. In Fig. 1, warehouse 1 sends products to retailers 1 and 2, each of which has a stochastic demand with means d_1 and d_2, respectively, and variances u_1 and u_2, respectively. Warehouse 2 supplies products to retailers 3, 4 and 5, and warehouse 3, to retailers 6 and 7. The operation of the warehouses incurs two type of cost: one is proportional to the average supplied demand (made up of holding and handling costs), and the other one is proportional to the standard deviation of supplied demand (due to the safety stock). Under constant lead times and levels of service, the safety stock is proportional to standard deviation of the supplied demand.

Thus, safety stock kept in warehouse 1, must be proportional to $\sqrt{u_1 + u_2}$; safety stock in the warehouse 2 must be proportional to $\sqrt{u_3 + u_4 + u_5}$, and the one in warehouse 3 must be proportional to $\sqrt{u_6 + u_7}$. Clearly the system's cost depends on the retailers assignment scheme. For instance, if we closed warehouse 1 and its clients were assigned to warehouse 2, significant cost changes occur. The fixed-installation cost of the warehouse 1 would be eliminated, and the transportation costs (from plant to warehouses and from warehouses to retailers) would change; safety stock cost would be reduced, because total safety stock kept on warehouse 2 would be

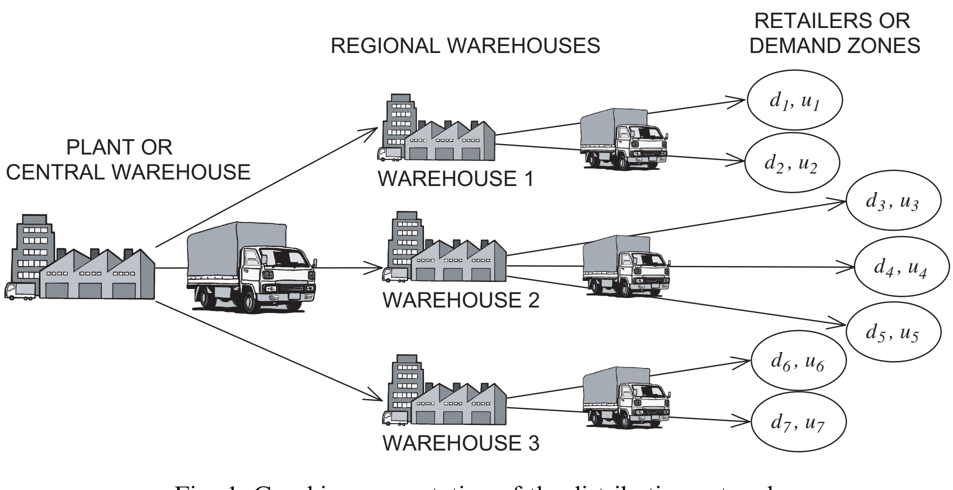

Fig. 1. Graphic representation of the distribution network.

proportional to $\sqrt{u_1+u_2+u_3+u_4+u_5}$ (clearly lower than $\sqrt{u_1+u_2}+\sqrt{u_3+u_4+u_5}$). This situation is known in the literature as risk pooling. This paper presents a non-linear-mixed-integer model to find an optimal configuration of network, considering the installation, transportation, ordering and holding, distribution network design with risk pooling effect model (DNDRP), along with a solution approach based on Lagrangian relaxation.

In the case analyzed in this paper, the plant location is known and fixed. Therefore, the transportation costs between the plant and the warehouses grouped into a single arc cost between warehouses and retailers. This situation can be easily modified, for a more general setting. The inventory policy at the plant is not modeled explicitly.

The model stated on this paper is an extension of the classical capacitated facility location problem, which is already NP-hard. Thus, if we use this model to solve a great instance (which are easy to find in the real world), any commercial package will take a lot of time to solve the model for these instances, or even will not be able to solve it. Then, is necessary to develop a solving approach. The numerical results developed in this paper consider a relative little instance, especially to compare the results obtained to solve optimally the model, with to solve heuristically this model.

The next section presents a literature review. In Section 3 the inventory control policy is analyzed and the objective function of the problem is developed. Section 4 presents the DNDRP model, which solve simultaneously facility location and inventory control decisions. Section 5, presents a solution approach to solve the DNDRP model. The solution approach is based on a combination of Lagrangian relaxation and the sub-gradient method. Section 6 reports the numerical results and their interpretation. Finally, Section 7 presents the conclusions and some future research lines.

2. Literature review

Facility location problems (FLP), which are typically used to design distribution networks, involves determining the sites to install resources, as well as the assignment of potential consumers to those resources. One example of FLP, is the location of manufacturing plants, the assignment of warehouses to these plants and finally the assignment of retailers to each warehouse. This family of problems typically assume a linear cost function and a set of deterministic demands for the customers considered. These assumptions avoid to model interactions between facility location and inventory control decision. More precisely the phenomenon widely studied and known as risk pooling, cannot be modeled with FLP.

Bramel et al. (2000) show three models to solve classical cases of FLP. The first one, called *P-Median Problem*, deals with the optimal location of P identical warehouses, for which there are M candidate sites. These warehouses must serve incoming orders from N retailers. This model does not consider installation costs, and the capacity of each warehouse is unlimited. The second problem is called capacitated facility location problem (CFLP). In this case the number of warehouses is variable and there are capacity constraints for each warehouse, as well as installation fixed costs. The third model, called distribution system design problem (DSDP), considers multiple plants with fixed capacities (the number and location of plants are fixed and known), and considers K different products in contrast with the single-product previous models. Daskin (1995) develops a similar and deeper review of logistics network design problems. These models are used to design distribution networks, but a common factor is that anyone include tactical decisions, as the inventory control, levels of service, vehicle routing decisions, etc. Thus, these models keeps the strategic issues, as facility location decisions, unlinked from tactical and operational issues.

Nozick (2001) presents a model based on the CFLP, which considers *covering* restrictions, i.e., a minimum demand to be *covered* by each warehouse. This criterion establishes that a customer is covered by a given warehouse, if and only if it is located within a given distance from that particular warehouse. This model allows to consider the level of service perceived by customers, in terms of time or distance, but does not consider holding cost and level of service in terms of the fulfilled effectively demand.

Melkote et al. (2001) show an integrated model for transportation network design and facility location problem. In addition to the standard aspects of the FLP, the model must identify the arcs on which to move products, considering transference nodes. In some practical cases, the installation of arcs is considered as a tactical decision, in contrast to the facility location decisions. Thus, this paper includes tactical decisions into a FLP, but inventory control decisions remains un-modeled.

In our paper we generalize the CFLP model, including simultaneously inventory control decisions. Furthermore, this approach can be easily applied into any FLP, as the models discussed in Bramel et al. (2000), Daskin (1995), Nozick (2001) and Melkote et al. (2001), among others.

In terms of inventory management, Winston (1997), Simchi-Levi et al. (2000), Bowersox et al. (1996), Anderson (1994) and Coyle et al. (1992), present the basic models for inventory control, on which the most used commercial-packages are based. In these works, the classic EOQ [1] model, and its typical variations (price discounting, continuous production or replenishment rate, etc.) are introduced. Porteus (1990) shows a selection of models based on the classic *newsvendor problem*, which considers a stochastic demand governed by a generic probability distribution function and a penalty cost for unfulfilled demand. He sets ordering quantities, which minimize the ordering, holding and unfulfilled demand costs. Furthermore, these models are developed only for a single period, considering initial stock and partial backlogging. These works, Winston (1997), Simchi-Levi et al. (2000), Bowersox et al. (1996), Anderson (1994), Coyle et al. (1992) and Porteus (1990), consider only a single location and, the demand and the level of service do not depend on the customers assigned to this location.

Cachon (2001) and Axsäter (2000) develop exact approaches to evaluate the system's cost for a two-echelon supply chain (one-warehouse-multi-retailer system), considering a periodic review and continuous review, respectively. These approaches consider the order quantities, the re-order-points, as fixed parameters. Ettl et al. (2000) analyze a complex network of supply processes, manufacturing stages and end consumers, determining re-order-points for each site of the network, given a set of service levels for each site that serves end consumers. This model minimizes the holding cost for the entire system and allows to determine the service levels, for a given set of re-order points. The sites are modeled as infinite-server queues, where the demand follows a compound Poisson process and the stochastic service-times are deduced trough the ordering process at each site. In Cachon (2001), Axsäter (2000) and Ettl et al. (2000), more general assumptions and more complex networks are considered, in contrast to the previous

[1] Economic order quantity.

works—Winston (1997), Simchi-Levi et al. (2000), Bowersox et al. (1996), Anderson (1994), Coyle et al. (1992), and Porteus (1990). But, the network cannot be modified, in terms of assignment of the customers and installation of the warehouses. Thus, interactions between facility location and inventory control decisions, among others similar interactions, cannot be modeled.

Simchi-Levi et al. (2000), Chen et al. (2000), Lee et al. (1997a,b) and Fransoo et al. (2000), investigate the distortion of demand at each stage of a supply chain, from the final consumers to upstream stages. This phenomenon is known as the *bullwhip effect*. The authors explain the causes and the consequences of the bullwhip effect, and describe some strategies to cope with this problem, as well as different methods for its measurement. Furthermore, Chen (1999a), Chen (1999b), Cachon et al. (2000), Lee et al. (2000), and Xu et al. (2000), investigate different aspects of the information management associated with the inventory control. These issues (the bullwhip effect and the levels of shared information) can be included into the DNDRP model, if we considered a greater number of stages, or if we explicitly modeled the inventory decisions at the plant or central warehouse. The latter would allows a comparison of a non-communicative network versus another one with different levels of information sharing. This incorporation is a straightforward extension of the DNDRP model.

3. Inventory control policy and total system cost

At any site or warehouse i, we assume a continuous inventory revision, and a (Q_i, RP_i) [2] policy to meet a stochastic demand pattern. Furthermore, we assume a stochastic demand with mean D_i (units of product per time unit) and variance U_i, for each warehouse i. We also consider that the plant takes a lead time of LT_i to fulfill an incoming order from warehouse i.

The evolution of the inventory level at warehouse i is showed in Fig. 2. Note that when the inventory level falls below RP_i, an order of Q_i units is triggered, which is received LT_i time units later. Once the order is placed, a difference appears between the on-hand-inventory (continuous line), and the inventory-position (segmented line). This difference vanishes when the units arrive at site i.

This policy does not penalize unfulfilled demands. Instead it sets a re-order-point RP_i. Once an order is submitted the inventory level should cover the demand produced during lead time LT_i, with a given probability $1-\alpha$. This probability is known as the *level of service* for the system. The level-of-service constrain can be expressed as follows:

$$\mathrm{Prob}(D(LT_i) \leqslant RP_i) = l - \alpha \tag{1}$$

where $D(LT_i)$ is the random demand during the lead time, at warehouse i. If we assume a Normally distributed demand, RP_i can be determined as follows:

$$RP_i = D_i \cdot LT_i + Z_{1-\alpha} \cdot SD_i \cdot \sqrt{LT_i} \tag{2}$$

where $Z_{1-\alpha}$ is the value of the Standard Normal distribution, which accumulates a probability of $1-\alpha$. This parameter is assumed fixed for the entire network, determining a uniform level of service for the system, which will be called K hereafter.

[2] i.e., a fixed quantity Q_i is ordered to the supplier, once the inventory level falls to or below a re-order-point RP_i.

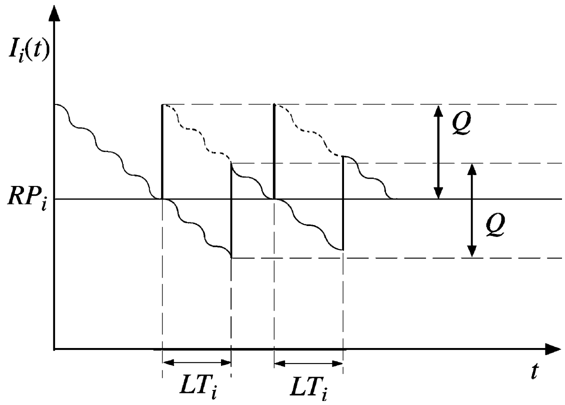

Fig. 2. Evolution of the inventory level $I_i(t)$ at site i.

Let HC_i be the holding cost per unit of product and time unit for warehouse i ($/unit-day). Then, the average holding cost rate for each warehouse i, based on expression (2), can be written as

$$HC_i \cdot K \cdot \sqrt{LT_i} \cdot \sqrt{U_i} + HC_i \cdot Q_i/2 \tag{3}$$

The first term in (3), is the average expenditure associated with safety stock kept at warehouse $i(K \cdot \sqrt{LT_i} \cdot \sqrt{U_i})$. The second term of the expression (3), is the average expenditure incurred due to the holding the order quantity Q_i, which is the inventory used to cover the demand arisen between two successive orders. Thus, if OC_i [3] is the ordering cost at site i, RC_i, is the transportation unit cost, between the plant and the warehouse i, and TP_i is the elapsed time between two consecutive orders for site i, the operation cost during this period is given by

$$RC_i \cdot Q_i + OC_i + \left(HC_i \cdot K \cdot \sqrt{LT_i} \cdot \sqrt{U_i} + HC_i \cdot Q/2\right) \cdot TP_i \tag{4}$$

Then, if we divide (4) by TP_i (which equals Q_i/D_i), the cost rate incurred at site i is given by the following expression:

$$(RC_i + OC_i/Q_i) \cdot D_i + HC_i \cdot K \cdot \sqrt{LT_i} \cdot \sqrt{U_i} + HC_i \cdot Q_i/2 \tag{5}$$

Now let us consider the following binary variables:

Z_i it takes the value 1, if a warehouse is installed on site i, and 0 otherwise
Y_{ij} it takes the value 1 if the warehouse on site i serves customer j, and 0 otherwise

If d_j is the average demand of retailer j, the total operational rate cost for the whole network can be written as

$$\sum_{i=1}^{N} \left(HC_i \cdot K \cdot \sqrt{LT_i} \cdot \sqrt{U_i} + HC_i \cdot Q_i/2\right) + \sum_{i=1}^{N} \sum_{i=1}^{M} (RC_i + OC_i/Q_i) \cdot d_j \cdot Y_{ij} \tag{6}$$

[3] This cost is incurred when an order is arisen, and it does not depend on the order size.

In the last expression, and for the rest of the paper, we assume the following evident relation:

$$\sum_{j=1}^{M} d_j \cdot Y_{ij} = D_i \quad \forall i = 1,\ldots,N \tag{7}$$

which relate the average demand served by the warehouses with the average demand of the customers. Furthermore, if TC_{ij} is the transportation unit cost between the warehouse i and the retailer j, the associated cost rate of the entire network is

$$\sum_{i=1}^{N}\sum_{j=1}^{M} TC_{ij} \cdot d_j \cdot Y_{ij} \tag{8}$$

Then, from expression (8) and the second term of the expression (6), the replenishment network's cost rate can be written as follows:

$$\sum_{i=1}^{N}\sum_{j=1}^{M}(TC_{ij} + RC_i + OC_i/Q_i) \cdot d_j \cdot Y_{ij} \tag{9}$$

Finally, lets consider the following parameters:

F_i installation fixed cost for warehouse i
TH planning horizon

Thus, if we consider the fixed installation cost for each warehouse, the total cost of system is

$$\sum_{i=1}^{N} F_i \cdot Z_i + TH \cdot \sum_{i=1}^{N} HC_i \cdot K \cdot \sqrt{LT_i} \cdot \sqrt{U_i} + TH \cdot \sum_{i=1}^{N} HC_i \cdot \frac{Q_i}{2} + TH \cdot \sum_{i=1}^{N}\sum_{j=1}^{M}\left(TC_{ij} + RC_i + \frac{OC_i}{Q_i}\right) \cdot d_j \cdot Y_{ij} \tag{10}$$

TH allows to sum the installation cost incurred at the begin of the planning horizon, and the rate cost incurred by the entire network. Furthermore, TH can be replaced by a factor which depend on, beside of the time period, an interest rate of capital. Even, if the fixed cost for each warehouse i, F_i, is a fixed cost rate, we can minimize the cost rate of the entire network, then we must consider $TH = 1$.

Finally, the objective function, given by the expression (10), is reformulated considering the optimization of the order quantity Q_i for each warehouse i. In this case we assume there is not capacity constraint for the order quantities. Thus, differentiating the objective function in terms of Q_i, for each $i = 1,\ldots,N$, and equaling to zero (minimizing the system cost in a centralized approach) we obtain:

$$TH \cdot \frac{HC_i}{2} - TH \cdot \frac{OC_i}{Q_i^2} \cdot \sum_{j=1}^{M} d_j \cdot Y_{ij} = 0 \tag{11}$$

From the Eq. (11) we obtain:

$$Q_i^* = \sqrt{\frac{2 \cdot OC_i \cdot D_i}{HC_i}} \quad \forall i = 1, \ldots, N \tag{12}$$

Note that the expression (12) corresponds to the same outcome of the classical EOQ model. This is an interesting results, because the isolated optimization gives the same outcome that the centralized optimization. However, there is a little aspect which must be noted. In the expressions (10) and (12) the ordering cost, OC_i, and the holding cost, HC_i, should correspond to the total ordering and holding costs of the system, including the expenditure incurred by the warehouses and by the plant, because they are incurred by the same owner. Furthermore, the expression (12) differ from static-sequential EOQ model, because the former depend of configuration of network, given by the variables Y_{ij} and Z_i, trough the variable D_i. Thus, if we modify the network, then, the optimal order quantity will change, similar to a best response function. Then, a bi-level problem is modeled simultaneously as a single-level problem.

Replacing (12) in the expression (10), the objective function can be expressed as follows:

$$\sum_{i=1}^{N} F_i \cdot Z_i + \sum_{i=1}^{N} \sum_{j=1}^{M} TH \cdot (TC_{ij} + RC_i) \cdot d_j \cdot Y_{ij} + \sum_{i=1}^{N} TH \cdot \sqrt{2.HC_i \cdot OC_i} \cdot \sqrt{D_i} + \sum_{i=1}^{N} TH \cdot HC_i \cdot K \cdot \sqrt{LT_i} \cdot \sqrt{U_i} \tag{13}$$

We must note that $\sqrt{2 \cdot HC_i \cdot OC_i} \cdot \sqrt{D_i}$ represents the optimal expenditure of the EOQ model, considering the ordering and holding costs.

4. Formulation of DNDRP model

In this section we present the model to solve the distribution network design with risk pooling problem, DNDRP, based on the inventory control policy stated in Section 3, and the objective function given by expression (13). This model is an extension of the capacitated facility location problem, CFLP, and consist of the determination of an optimal configuration of the distribution network, taking inventory decisions and the associated cost, into account. We assume there is a single owner of the network, who is responsible of the entire cost of the system. This assumption is consistent with a practical case studied, given by the distribution system of a firm which distributes and commercializes frozen food in Chile.

The DNDRP model, should decide where to install warehouses, with N potential sites, to serve a set of M retailers. Each retailer must be served by only one-warehouse. The model must locate warehouses and assign retailers to them, considering capacity constraints. Beside of the parameters and the variables defined in Section 3, let us define the following parameters:

Cap_i capacity at warehouse i
u_j variance of the demand per time unit for customer or retailery j

The DNDRP model is the following:

$$\text{Min} \quad \sum_{i=1}^{N} F_i \cdot Z_i + \sum_{i=1}^{N} \sum_{j=1}^{M} TH \cdot (TC_{ij} + RC_i) \cdot d_j \cdot Y_{ij} + \sum_{i=1}^{N} TH \cdot \sqrt{2.HC_i \cdot OC_i} \cdot \sqrt{D_i} + \sum_{i=1}^{N} TH \cdot HC_i \cdot K \cdot \sqrt{LT_i} \cdot \sqrt{U_i} \tag{14}$$

$$\text{subject to:} \quad \sum_{i=1}^{N} Y_{ij} = l \quad \forall j = 1, \ldots, M \tag{15}$$

$$\sum_{j=1}^{M} d_j \cdot Y_{ij} \leqslant \text{Cap}_i \cdot Z_i \quad \forall i = 1, \ldots, N \tag{16}$$

$$\sum_{j=1}^{M} d_j \cdot Y_{ij} = D_i \quad \forall i = 1, \ldots, N \tag{17}$$

$$\sum_{j=1}^{M} Y_{ij} \cdot u_j = U_i \quad \forall i = 1, \ldots, N \tag{18}$$

$$Z_i, Y_{ij} \in \{0.1\} \quad \forall i, \ldots, N, \forall j, \ldots, M \tag{19}$$

Eq. (15) assures that each retailer is served exactly by one warehouse. Eq. (16) assures that the capacity of the warehouses is not exceeded (only if the warehouse is installed). Eq. (17) computes the served average demand by warehouse i. Eq. (18) computes the total variance of served demand by warehouse i. Implicitly we assume that the demands are independently distributed across the retailers, thus all the covariance terms are zero. Finally (19) states integrality for the variables Y_{ij} and Z_i.

We must note that the DNDRP model is NP-hard, because it is an extension of the CFLP model, which is already NP-hard. In addition, the objective function is non-linear, resulting in a model which is hard to solve, especially for great instances. Thus an heuristic approach to solve this problem must be developed.

5. Solution approach

This section describes the approach used to solve the DNDRD model, which is based on Lagrangian relaxation and the sub-gradient method. The cost parameters are expressed as follows:

$$C_{ij} = TH \cdot (RC_i + TC_{ij}) \cdot d_j, \quad CL_i = TH \cdot \sqrt{2.HC_i \cdot OC_i}, \quad CS_i = TH \cdot HC_i \cdot K \cdot \sqrt{LT_i}$$

Then, the objective function can be written as follows:

$$\sum_{i=1}^{N} F_i \cdot Z_i + \sum_{i=1}^{N} \sum_{j=1}^{M} C_{ij} \cdot Y_{ij} + \sum_{i=1}^{N} CL_i \cdot \sqrt{D_i} + \sum_{i=1}^{N} CS_i \cdot \sqrt{U_i} \tag{20}$$

5.1. A stronger formulation and Lagrangian relaxation of DNDRP

We proposes a stronger formulation for the DNDRP, incorporating two additional constraints into the model, given by

$$\sum_{i=1}^{N} D_i \leqslant DT = \sum_{j=1}^{M} d_j \tag{21}$$

$$\sum_{i=1}^{N} U_i \leqslant VT = \sum_{j=1}^{M} u_j \tag{22}$$

Expression (21) assures that the average total demand assigned to the warehouses does not exceed the total average demand of the retailers. Eq. (22) guarantees that the variance of the demand assigned to the warehouses does not exceed the total variance of the demand of the retailers. These constraints are redundant, because they correspond (on equality) to the summation of (17) and (18), respectively, over all the warehouses. However they are not redundant for the sub-problems obtained from the relaxation proposed afterward. Furthermore, constraints (17) and (18) are replaced by

$$\sum_{j=1}^{M} Y_{ij} \cdot d_j \leqslant D_i \quad \forall i = 1, \ldots, N \tag{23}$$

$$\sum_{j=1}^{M} Y_{ij} \cdot u_j \leqslant U_i \quad \forall i = 1, \ldots, N \tag{24}$$

Note that constraints (23) and (24) reach equality at optimality—for fixed values of Y_{ij} and Z_i, it is always better to reduce D_i and U_i. Thus, the new constraints (21) and (22) are also satisfied on equality at optimality. Then, the changes on constraints (17) and (18) do not alter the feasibility and optimality of the problem, but they are helpful for the relaxation scheme to be introduced in Section 5.3.

We propose to relax the constraints (17) and (18), which link D_i and U_i (continuous variables) with Z_i and Y_{ij} (integer variables). Then, associating the dual variables ω and λ to the constraints (17) and (18), respectively, the model can be written as follows:

$$\begin{aligned} \theta(\omega, \lambda) = \quad \text{Min} \quad & \sum_{i=1}^{N}\left(CL_i \cdot \sqrt{D_i} - \omega_i \cdot D_i\right) + \sum_{i=1}^{N}\left(CS_i \cdot \sqrt{U_i} - \lambda_i \cdot U_i\right) + \sum_{i=1}^{N} F_i \cdot Z_i \\ & + \sum_{i=1}^{N} \sum_{j=1}^{M} (C_{ij} + \omega_i \cdot d_j + \lambda_i \cdot u_i) \cdot Y_{ij} \\ \text{subject to:} \quad & (15),\ (16),\ (19),\ (21) \text{ and } (22) \end{aligned} \tag{25}$$

The optimization algorithm must solve the relaxed problem for a given set of Lagrangian multipliers in an iterative structure, which is explained in the next section.

5.2. Sub-problems solving

For given values of the vectors ω and λ at iteration k, ω^k and λ^k, $\theta(\omega^k, \lambda^k)$ can be decomposed into three sub-problems. The first two sub-problems are the following:

$$\begin{aligned} \mathrm{SP1}^k \quad \text{Min} \quad & \sum_{i=1}^{N} \left(CL_i \cdot \sqrt{D_i} - \omega_i^k \cdot D_i \right) \\ \text{subject to:} \quad & \sum_{i=1}^{N} D_i \leqslant DT \\ & D_i \geqslant 0 \quad \forall i = 1, \ldots, N \end{aligned} \tag{26}$$

$$\begin{aligned} \mathrm{SP2}^k \quad \text{Min} \quad & \sum_{i=1}^{N} \left(CS_i \cdot \sqrt{U_i} - \lambda_i^k \cdot U_i \right) \\ \text{subject to:} \quad & \sum_{i=1}^{N} U_i \leqslant VT \\ & U_i \geqslant 0 \quad \forall i = 1, \ldots, N \end{aligned} \tag{27}$$

Solving $\mathrm{SP1}^k$ and $\mathrm{SP2}^k$, implies to search the warehouse r, that gives the minimum value for $CL_r \cdot \sqrt{DT} - \omega_r^k \cdot DT$, and the warehouse w, that gives the minimum value for $CS_w \cdot \sqrt{VT} - \lambda_w^k \cdot VT$. If the values of these expressions were negative, the optimal solution can be obtained assigning the following value (demonstration is in Appendix A):

$$D_i^k = \begin{cases} DT & i = r \\ 0 & i \neq r \end{cases} \quad \text{and} \quad U_i^k = \begin{cases} VT & i = w \\ 0 & i \neq w \end{cases} \tag{28}$$

If $CL_r \cdot \sqrt{DT} - \omega_r^k \cdot DT \geqslant 0$, then $U_i^k = 0$ for each warehouse i, and if $CS_w \cdot \sqrt{VT} - \lambda_w^k \cdot VT \geqslant 0$, $D_i^k = 0$ for each warehouse i.

The third sub-problem is more cumbersome, though it is a widely studied problem, and it is a CFLP. This sub-problem is

$$\mathrm{SP3}^k \quad \text{Min} \quad \sum_{i=1}^{N} F_i \cdot Z_i + \sum_{i=1}^{N} \sum_{j=1}^{M} (C_{ij} + \omega_i^k \cdot d_j + \lambda_i^k \cdot u_j) \cdot Y_{ij} \tag{29}$$

$$\text{subject to:} \quad \sum_{j=1}^{M} d_j \cdot Y_{ij} \leqslant \mathrm{Cap}_i \cdot Z_i \quad \forall i = 1, \ldots, N \tag{30}$$

$$\sum_{i=1}^{N} Y_{ij} = 1 \quad \forall j = 1, \ldots, M \tag{31}$$

$$Y_{ij}, Z_i \in \{0, 1\} \quad \forall i = 1, \ldots, N, \ \forall j = 1, \ldots, M \tag{32}$$

$\mathrm{SP3}^k$, or CFLP, can be solved through a myriad of different methods—see Bramel et al. (2000), Daskin (1995), Nozick (2001) and Melkote et al. (2001). Once this problem is solved, the primal optimal values Z_i^k and Y_{ij}^k, for the kth iteration are known.

5.3. Update the Lagrangian multipliers

Once the sub-problems are solved, we must update the values of the dual variables, ω_i^k and λ_i^k, to augment the value of $\theta(\omega, \lambda)$. The updating procedure selected here is the sub-gradient method, which considers violation-vectors, VD^k and VU^k, as the ascending direction. The components of the violation-vectors can be obtained as follows:

$$\left.\begin{aligned} VD_i^k &= \sum_{j=1}^{M} Y_{ij}^k \cdot d_j - D_i^k \\ VU_i^k &= \sum_{j=1}^{M} Y_{ij}^k \cdot u_j - U_i^k \end{aligned}\right\} \quad \forall i = 1, \ldots, N \tag{33}$$

The steps size for iteration k, are determined by

$$\alpha_\omega^k = \rho^k \frac{(Z_k^{\text{Sup}} - Z_k^{\text{Inf}})}{\|VD^k\|^2} \quad \text{and} \quad \alpha_\lambda^k = \rho^k \frac{(Z_k^{\text{Sup}} - Z_k^{\text{Inf}})}{\|VU^k\|^2} \tag{34}$$

where Z_k^{Sup} is an upper bound for the primal problem, and Z_k^{Inf} is a lower bound for the dual problem—which are available on iteration k. Furthermore, ρ^k is a control parameter which must be stated from an empirical analysis, and typically satisfies $0 < \rho^k < 2$. Thus, the equations for updating the Lagrangian variables are

$$\omega_i^{k+1} = \text{Max}\{0, \omega_i^k + \alpha_\omega^k \cdot VD_i^k\} \quad \text{and} \quad \lambda_i^{k+1} = \text{Max}\{0, \lambda_i^k + \alpha_\lambda^k \cdot VU_i^k\} \tag{35}$$

The procedure to obtain Z_k^{Sup} (known as the Lagrangian heuristic) consists, in to calculate a feasible solution for the primal problem, which, in this case, is based on the solution stated for SP3^k, considering:

$$\overline{D}_i^k = \sum_{j=1}^{M} Y_{ij}^k \cdot d_j \quad \text{and} \quad \overline{U}_i^k = \sum_{j=1}^{M} Y_{ij}^k \cdot u_j \tag{36}$$

In consequence, Z_k^{Sup} is calculated as

$$Z_k^{\text{Sup}} = \text{Min}\{Z_{k-1}^{\text{Sup}}, \overline{Z}_k\} \tag{37}$$

where

$$\overline{Z}_k = \sum_{i=1}^{N} CL_i \cdot \sqrt{\overline{D}_i^k} + \sum_{i=1}^{N} CS_i \cdot \sqrt{\overline{U}_i^k} + \sum_{i=1}^{N} F_i \cdot Z_i^k + \sum_{i=1}^{N} \sum_{j=1}^{M} C_{ij} \cdot Y_{ij}^k \tag{38}$$

On the other hand, Z_k^{Inf} is calculated through the next expression:

$$\begin{aligned} Z_k^{\text{Inf}} = {} & \sum_{i=1}^{N} \left(CL_i \cdot \sqrt{D_i^k} - \omega_i^k \cdot D_i^k \right) + \sum_{i=1}^{N} \left(CS_i \cdot \sqrt{U_i^k} - \lambda_i^k \cdot U_i^k \right) + \sum_{i=1}^{N} F_i \cdot Z_i^k \\ & + \sum_{i=1}^{N} \sum_{j=1}^{M} (C_{ij} + \omega_i^k \cdot d_j + \lambda_i^k \cdot u_j) \cdot Y_{ij}^k \end{aligned} \tag{39}$$

The algorithm keeps running until some convergence criterion is met (see Appendix B).

6. Results and discussion

In this section we show the main results obtained from applying the DNDRP to a numerical case. This application of DNDRP incorporates the choice of two different capacity levels for each warehouse—one decision is whether to install it or not, and the other one is choosing how much capacity to allocate. Considering more than two capacity levels is a straightforward extension. The capacity levels are modeled considering the constraints (40), where Z_i^M and Z_i^H are binary variables, taking the value 1 if a warehouse is installed on site i with medium (CapM_i) or high capacity (CapH_i), respectively, and 0 otherwise. The fixed installation costs are CfixM_i and CfixH_i, respectively. Furthermore, we must assure for each site i, that only one variable of Z_i^M and Z_i^H be equal to 1 ($Z_i^H + Z_i^M \leqslant 1, \forall i = 1, \ldots, N$).

$$\sum_{j=1}^{M} Y_{ij} \cdot d_j \leqslant Z_i^M \cdot \text{CapM}_i + Z_i^H \cdot \text{CapH}_i \quad \forall i = 1, \ldots, N \tag{40}$$

We consider 10 potential warehouse locations, 20 customers, and different sets of parameters generating 25 cases. Tables 1–3, show the parameters values of the base case (Case 0), which are then varied to obtain the other cases.

The nomenclature is as follows: xVC represents a (+ or −) x% change in the value of the demand's variation coefficient, with respect to the base case; xHC represents a (+or −) x% change in the value of the parameter HC_i, with respect to the base case; finally, Y_xHC, represents a (+ or −) x% change in the value of the parameter HC_i, for the Case Y (being Y one of the changes in the demand's variation coefficient explained above). For all cases, we consider $K = 1.96$, 1.28, 0.67, and 0 (associated with 97.5%, 90%, 75% and 50% for the level of service) and $TH = 1000$ (planning horizon). Any sensitivity analysis must give similar results if we changed other parameters (e.g. lead times, level of service, or ordering cost), since they would change the same cost factors.

We compare the results obtained from the DNDRP (including the choice of two different capacities) and the results obtained from the classical CFLP as a benchmark, within a sequential approach, SDND—sequential distribution network design. The SDND approach consists in to evaluate the configuration found by CFLP (which only considers the installation and transportation cost) with the objective function of the DNDRP—Eq. (13). Note that CFLP is equivalent to the DNDRP, when $HC_i = 0$, for $i = 1, \ldots, N$, where the variance of the demand does not affect

Table 1
Parameters of the base case, associated with warehouses W-i

Parameter	W-1	W-2	W-3	W-4	W-5	W-6	W-7	W-8	W-9	W-10
CfixM	2,391,084	1,722,391	1,925,025	2,107,396	1,722,391	2,107,396	1,864,235	2,127,660	1,985,816	2,046,606
CfixH	3,347,518	2,411,348	2,695,035	2,950,355	2,411,348	2,950,355	2,609,929	2,978,723	2,780,142	2,865,248
CapM	47	49	47	42	46	46	46	49	49	44
CapH	72	75	82	70	80	63	76	73	82	78
HC	184	133	148	163	133	163	144	164	153	158
OC	846	757	810	767	669	869	686	786	656	654
LT	4	3	3	3	2	4	2	3	2	2
RC	62	46	55	47	30	66	33	51	28	27

Table 2
Parameters of the base case, demand of customers C-j

Customer	C-1	C-2	C-3	C-4	C-5	C-6	C-7	C-8	C-9	C-10
Average	10	15	16	15	12	18	17	11	14	14
Variance	108	221	295	268	135	323	306	117	166	194
VC	0.962	1.009	0.932	0.916	1.033	1.002	0.972	1.017	1.087	1.005
Customer	C-11	C-12	C-13	C-14	C-15	C-16	C-17	C-18	C-19	C-20
Average	15	12	19	18	19	14	13	14	10	12
Variance	212	139	405	287	402	233	141	207	105	139
VC	1.030	1.018	0.944	1.063	0.948	0.917	1.095	0.973	0.976	1.018

Table 3
Parameters of the base case, assignment cost between warehouse W-i and customer C-j

TC_{ij}	W-1	W-2	W-3	W-4	W-5	W-6	W-7	W-8	W-9	W-10
C-1	435	234	90	483	195	603	333	396	300	411
C-2	250	128	120	260	118	350	196	242	148	224
C-3	243	243	264	159	166	172	120	170	151	46
C-4	342	396	468	200	352	64	268	300	306	174
C-5	455	322	262	427	112	440	152	177	325	305
C-6	111	70	168	175	175	266	216	263	65	180
C-7	164	86	165	174	144	257	195	243	74	155
C-8	158	125	300	275	332	447	384	474	144	310
C-9	186	51	177	274	255	417	323	379	158	300
C-10	381	385	402	229	287	160	171	216	278	130
C-11	94	132	238	178	266	304	296	356	98	226
C-12	397	327	330	285	167	295	57	145	267	160
C-13	331	265	238	257	131	244	77	33	235	168
C-14	158	185	231	71	185	156	176	211	93	85
C-15	306	206	115	314	107	345	181	176	221	251
C-16	177	124	212	188	192	285	220	297	47	184
C-17	486	353	253	436	143	473	189	182	346	332
C-18	282	122	94	293	122	379	231	272	184	257
C-19	384	264	258	363	117	423	192	279	237	228
C-20	412	220	70	462	212	552	330	372	307	400

the total systems cost. Furthermore, evaluating the solution found by CFLP into the Eq. (13), is equivalent to solve sequentially the inventory decisions: the re-order-point and the order quantities are determined for each installed warehouse, i.e., considering the configuration found by CFLP as fixed. It is clear that the SDND will give a sub optimal solution (i.e., a more expensive network); the question we need to answer is under what circumstances the solution found by the DNDRP is significantly better than the simpler one, found by the SDND, i.e., what is the effect of demand variability, the magnitude of the holding costs, or the level of service.

We present two sets of results: those obtained by solving the DNDRP directly trough Branch & Bound with LINGO 6.0, and using the heuristic presented in Section 5 (DNDRP-LINGO and DNDRP-LR, respectively, in figures and tables). The first comparison (DNDRP-LINGO versus

SDND) denotes the contribution of the model in contrast to use the classical sequential approach, and the second comparison (DNDRP-LR versus SDND) denotes the potential benefit that can be obtained by using the heuristic approach, instead of the exact approach, in contrast to use SDND. The comparisons are presented in Tables 4–7. These tables show the objective function value

Table 4
Results of applying SDND and DNDRP, considering 50% for the level of service

Solutions summarize	SDND	DNDRP-LINGO		DNDRP-LR	
	Total cost	Total cost	% Save	Total cost	% Save
Case X_−50HC	600,891	600,891	0.00	605,891	−0.83
Case X_−25HC	624,418	624,418	0.00	624,521	−0.02
Case X	644,260	639,400	0.75	644,260	0.00
Case X_+25HC	661,725	653,912	1.18	661,725	0.00
Case X_+50HC	677,493	667,009	1.55	677,297	0.03

Table 5
Results of applying SDND and DNDRP, considering 75% for the level of service

Solutions summarize	SDND	DNDRP-LINGO		DNDRP-LR	
	Total cost	Total cost	% Save	Total cost	% Save
Case −50VC_−50HC	638,634	634,749	0.61	638,634	0.00
Case −25VC_−50HC	657,397	651,814	0.85	657,062	0.05
Case −25HC	676,273	666,040	1.51	675,319	0.14
Case 25VC_−50HC	695,100	681,690	1.93	685,682	1.35
Case 50VC_−50HC	713,981	701,800	1.71	702,577	1.60
Case −50VC_−25HC	681,078	670,022	1.62	681,078	0.00
Case −25VC_−25HC	709,244	700,592	1.22	707,435	0.26
Case −25HC	737,581	716,999	2.79	723,870	1.86
Case 25VC_−25HC	765,844	745,464	2.66	749,149	2.18
Case 50VC_−25HC	794,189	767,568	3.35	774,510	2.48
Case −50VC	719,798	702,188	2.45	719,552	0.03
Case −25VC	757,348	733,394	3.16	754,136	0.42
Case 0	795,126	764,802	3.81	776,477	2.35
Case 25VC	832,805	803,743	3.49	809,202	2.83
Case 50VC	870,593	833,453	4.27	843,010	3.17
Case −50VC_25HC	756,168	741,875	1.89	753,221	0.39
Case −25VC_25HC	803,116	780,789	2.78	782,800	2.53
Case −50HC	850,348	817,927	3.81	825,064	2.97
Case 25VC_25HC	897,458	854,978	4.73	889,787	0.85
Case 50VC_25HC	944,704	892,124	5.57	913,742	3.28
Case −50VC_50HC	790,842	770,508	2.57	786,917	0.50
Case −25VC_50HC	847,188	815,821	3.70	822,344	2.93
Case −50HC	903,876	860,402	4.81	873,417	3.37
Case 25VC_50HC	960,417	904,876	5.78	923,637	3.83
Case 50VC_50HC	1,017,120	949,467	6.65	974,373	4.20

Table 6
Results of applying SDND and DNDRP considering 90% for the level of service

Solutions summarize	SDND	DNDRP-LINGO		DNDRP-LR	
	Total cost	Total cost	% Save	Total cost	% Save
Case −50VC_−50HC	672,670	663,049	1.43	671,856	0.12
Case −25VC_−50HC	708,352	699,400	1.26	707,888	0.07
Case −25HC	744,250	722,562	2.91	74,1043	0.43
Case 25VC_−50HC	780,055	752,327	3.55	761,681	2.36
Case 50VC_−50HC	815,963	785,060	3.79	793,812	2.71
Case −50VC_−25HC	732,091	720,178	1.63	729,556	0.35
Case −25VC_−25HC	785,616	765,311	2.58	781,257	0.55
Case −25HC	839,465	808,550	3.68	835,998	0.41
Case 25VC_−25HC	893,174	850,793	4.75	863,375	3.34
Case 50VC_−25HC	947,038	893,145	5.69	937,352	1.02
Case −50VC	787,831	767,490	2.58	783,668	0.53
Case −25VC	859,201	818,078	4.79	832,812	3.07
Case 0	931,003	880,998	5.37	898,893	3.45
Case 25VC	1,002,618	937,329	6.51	961,110	4.14
Case 50VC	1,074,440	993,807	7.50	1,025,370	4.57
Case −50VC_25HC	841,205	810,732	3.62	835,462	0.68
Case −25VC_25HC	930,426	880,938	5.32	921,643	0.94
Case −50HC	1,020,187	951,525	6.73	976,947	4.24
Case 25VC_25HC	1,109,715	1,021,945	7.91	1,057,050	4.75
Case 50VC_25HC	1,199,501	1,092,548	8.92	1,132,410	5.59
Case −50VC_50HC	892,877	851,702	4.61	888,706	0.47
Case −25VC_50HC	999,946	935,946	6.40	959,581	4.04
Case −50HC	1,107,664	1,020,647	7.86	1,055,350	4.72
Case 25VC_50HC	1,215,102	1,105,147	9.05	1,147,940	5.53
Case 50VC_50HC	1,322,849	1,189,867	10.05	1,240,400	6.23

reached by SDND, the objective function value and the cost reduction (compared with SDND), reached by DNDRP-LINGO and DNDRP-LR, for each case considering in the sensitivity analysis, and considering different values for the level of service (50%, 75%, 90% and 97.5%).

The costs reduction given by DNDRP-LINGO compared with SDND, are summarized in Figs. 3 and 4 (considering only 75%, 90% and 97.5% for the level of service). The costs reduction average in Fig. 3, for each value of the sensitivity in holding costs, and for each value of the level of service, was obtained considering the cost reduction observed for each value of the sensitivity in variation coefficient. In Fig. 4, in contrast, the costs reduction average, for each value of the sensitivity in variation coefficient, and for each value of the level of service, was obtained considering the cost reduction observed for each value of the sensitivity in the holding cost.

We must note that the CFLP, in terms of the variables Z_i, and Y_{ij}, gives the same result for all the cases. This is so because it assumes deterministic demand and it neglects the inventory costs, hence its result does not change if we modify the holding cost or the variability of demand.

In Figs. 3 and 4 (which only consider 75%, 90% and 97.5% for the level of service), we observe clearly that the cost reduction reached by the DNDRP-LINGO is greater when the demand

Table 7
Results of applying SDND and DNDRP considering 97.5% for the level of service

Solutions summarize	SDND	DNDRP-LINGO		DND-LR	
	Total cost	Total cost	% Save	Total cost	% Save
Case −50VC_−50HC	710,677	694,640	2.26	710,173	0.07
Case −25VC_−50HC	765,253	750,429	1.94	761,331	0.51
Case −25HC	820,159	792,872	3.33	814,437	0.70
Case 25VC_−50HC	874,922	835,949	4.45	846,521	3.25
Case 50VC_−50HC	929,843	879,138	5.45	899,983	3.21
Case −50VC_−25HC	789,112	768,936	2.56	786,916	0.28
Case −25VC_−25HC	870,982	833,355	4.32	843,183	3.19
Case −25HC	953,348	898,125	5.79	943,444	1.04
Case 25VC_−25HC	1,035,501	962,741	7.03	989,507	4.44
Case 50VC_−25HC	1,117,890	1,027,525	8.08	1,063,010	4.91
Case −50VC	863,872	828,161	4.13	860,043	0.44
Case −25VC	973,042	914,062	6.06	935,273	3.88
Case 0	1,082,874	1,000,429	7.61	1,032,900	4.61
Case 25VC	1,192,421	1,086,592	8.88	1,130,880	5.16
Case 50VC	1,302,283	1,172,978	9.93	1,218,510	6.43
Case −50VC_25HC	936,232	885,452	5.42	927,363	0.95
Case −25VC_25HC	1,072,691	992,827	7.45	1,032,390	3.76
Case −50HC	1,209,977	1,100,785	9.02	1,140,540	5.74
Case 25VC_25HC	1,346,907	1,208,486	10.28	1,262,940	6.23
Case 50VC_25HC	1,484,231	1,316,468	11.30	1,374,790	7.37
Case −50VC_50HC	1,006,920	941,386	6.51	999,905	0.70
Case −25VC_50HC	1,170,679	1,070,241	8.58	1,111,680	5.04
Case −50HC	1,335,431	1,189,911	10.90	1,251,530	6.28
Case 25VC_50HC	1,499,756	1,329,045	11.38	1,388,640	7.41
Case 50VC_50HC	1,664,553	1,458,628	12.37	1,531,880	7.97

variability, the holding costs or the level of service, increase, especially due to the reduction of inventory cost and the number of warehouses installed. For example, for +50% in the sensitivity of the holding cost and a level of services of 97.5%, we observe a cost reduction average about 10%. Furthermore, when the sensitivity in variation coefficient is +50%, and for the same level of service we observe a cost reduction average about 9%. In contrast, for a −50% in the sensitivity of the holding cost and a level of service of 75%, we observe a cost reduction average about 1%, and for a −50% in the sensitivity of the variation coefficient, for the same level of service, we observe a similar cost reduction average. The latter result denotes the *risk pooling* phenomenon, and denotes the contribution of the simultaneous model, DNDRP, to design the distribution network.

On the other hand, if we considered the heuristic algorithm introduced in Section 5, which can be necessary for a greater instances, in contrast to the instances considered in this paper, we will reach a lower cost reduction. These comparisons are summarized in Figs. 5 and 6, for a level of service values of 75%, 90% and 97.5%. We observe, for these values of level of service, that the costs reduction average, are always greater than zero, assuring that the solutions found by

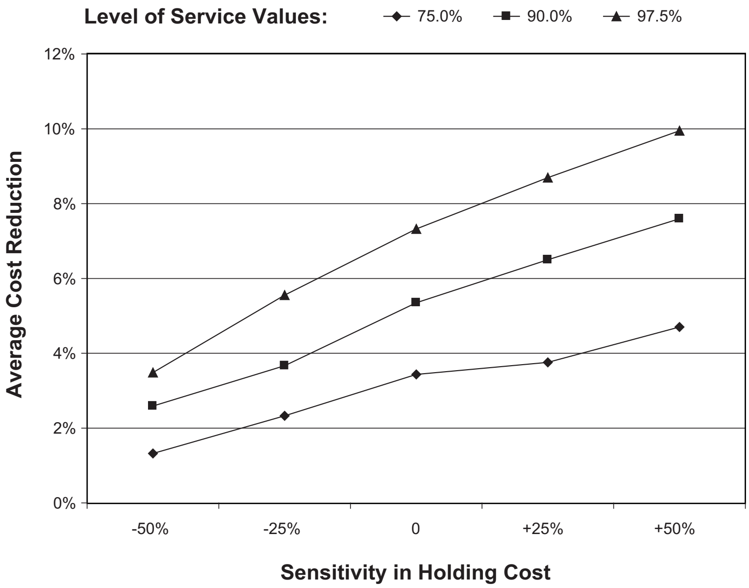

Fig. 3. Average cost reduction, comparing SDND versus DNDRP solved using LINGO.

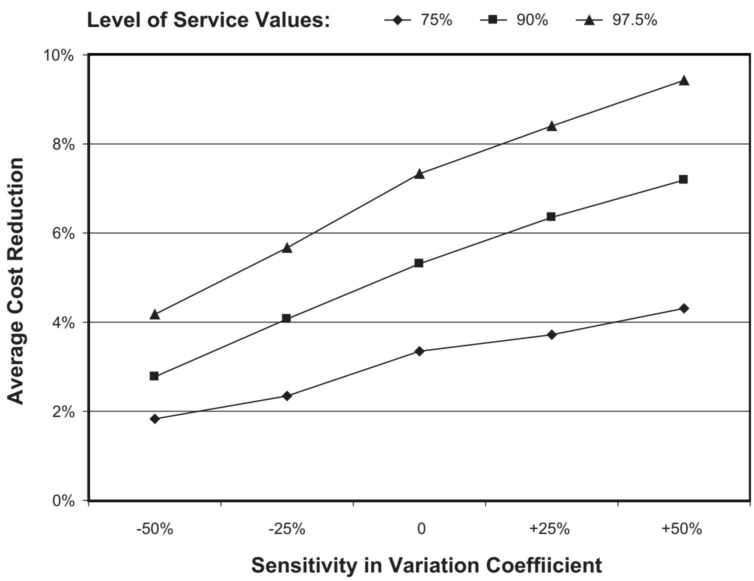

Fig. 4. Average cost reduction, comparing SDND versus DNDRP solved using LINGO.

DNDRP-LR (considering 75%, 90% and 97.5% for the level of service), are actually better than the benchmark solution, which is found by SDND.

Finally, if we use a re-order point equivalent to expected demand during lead times, i.e., $K = 0$ (which is equivalent to assume a deterministic demand, or to consider a 50% for the level of

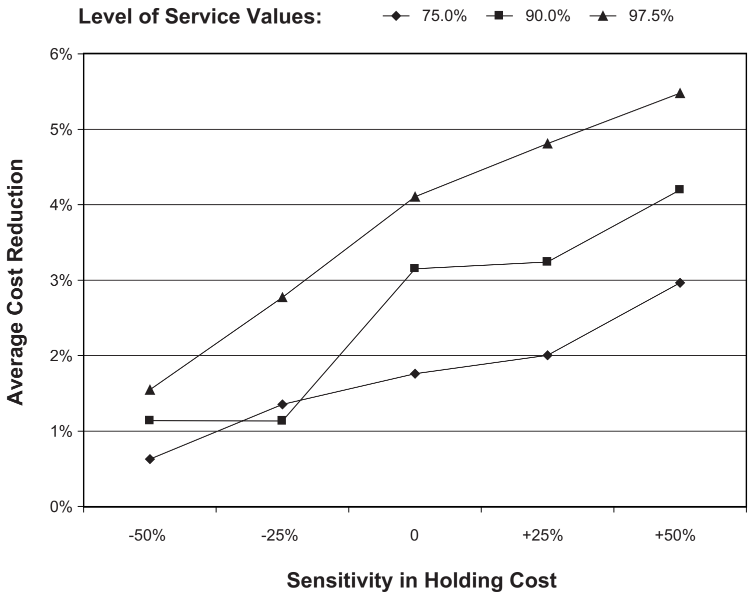

Fig. 5. Average cost reduction, comparing SDND versus DNDRP, solved using Lagrangian relaxation.

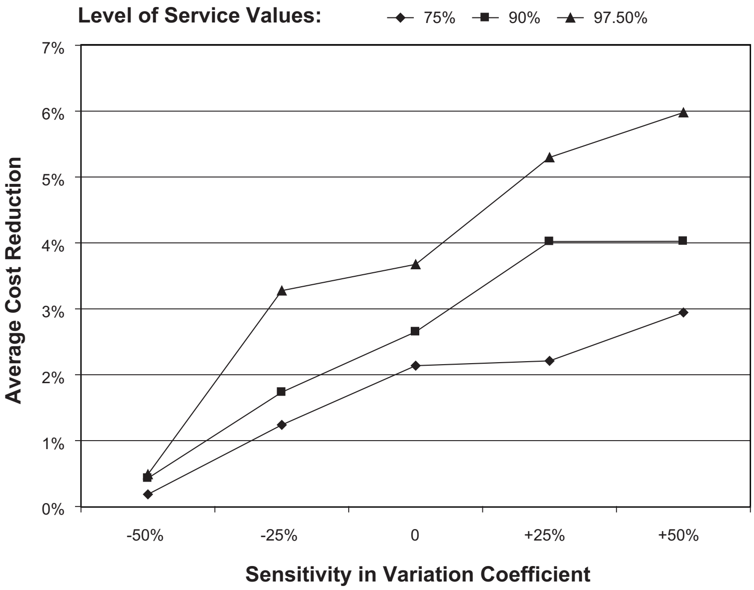

Fig. 6. Average cost reduction, comparing SDND versus DNDRP, solved using Lagrangian relaxation.

service), the model is indifferent of the variation coefficient. But, we must consider that the cost factor associated with D_i, $TH \cdot \sqrt{2 \cdot HC_i \cdot OC_i}$, is dependent of holding cost, and the augmenting of cost reduction in terms of holding costs hold. Thus, this term in the objective function denotes another interaction between facility location and inventory control decisions (beside of risk pooling), which is not considered in typical FLP's. If we used more warehouses, then, we obtain a more expensive network in terms of the ordering and the holding cost (assuming a deterministic demand). In Table 4, we observe the results of considering a level of service of 50%. It is observed, the cost reduction given by DNDRP-LINGO, is lower (even being zero) for the lower values of holding costs. However, if the holding costs are greater, the cost reduction are greater and could be relevant. This results hold for DNDRP-LR, although in a lower magnitude. Even, for the lowest-holding-cost cases, DNDRP-LR solution is worst than the SDND solution.

Acknowledging the fact that the computation time depends on an enormous number of factor and that it cannot be transferred from one experience to another, it is illustrative to report that in all the simulated cases, the DNDRP-LR always found the solution in less than 1 min; LINGO instead took about 5 min in average to solve these instances. These results suggest that the complexity of the problem precludes its exact solution when the problem size is higher.

We must remark that the sub-problem $SP3^k$, showed in Section 5.1, was not solved with the most efficient methods but rather with a general purpose subroutine in LINGO (Branch & Bound). This condition should affect the performance of the LR heuristic and it could be improved in further research—Bramel et al. (2000) and Daskin (1995) present different procedures that might improve the LR heuristic performance if properly implemented. Furthermore, the step-size parameter of the sub-gradient method, the convergence criterion (see Appendix B), and the control parameter's updating (see Section 5.2), were not "optimized" in the sense of using the most efficient implementation available in the literature, but rather using a straightforward implementation, possibly resulting in a non-efficient algorithm in terms of computation time.

7. Conclusions and future research

In the model introduced in our paper, DNDRP, which is used to design the distribution network, we optimize the magnitude for the orders of warehouses to plant. In this optimization, is interesting to note that the isolated optimization of the order quantities, gives the same results of the centralized optimization of them. The difference between the centralized and the isolated optimization, consists in the cost parameters considered (ordering and holding cost). The isolated optimization considers the cost parameters of the warehouses, while the centralized one, considers the cost parameters associated with the entire system.

We observe, in the numerical application discussed in Section 6, that the total cost reduction is higher as the holding cost, ordering cost, lead times and/or level of service (measured as the probability of satisfying all the demands) increase. These elements generate the expenditure in inventory management, and can be modified by the decision makers within a supply chain. In addition, when the variability of demand increases, the DNDRP achieves higher cost reductions. Note that the complexity of the analyzed problem precludes the general use of exact algorithms, especially for large size networks. Under these scenarios, the proposed LR heuristic offers an efficient solution method.

We conclude that in supply chains where the products are perishable or of high value, as the frozen food among others, the simultaneous approach (considering DNDRP model along with the LR heuristic) appears as a valuable (and easy to work with) tool for assisting the decision makers in the hard task of designing a distribution network.

Furthermore, the time required to solve the DNDRP using the LR heuristic can be significantly improved if we considered a more efficient procedure to solve $SP3^k$ at each iteration. For example, if we simultaneously relax the constraints associated with capacity at each warehouse (constraints (30) and/or (40)), and we develop a procedure more complex to search a feasible solution on each step, we can improve the heuristic algorithm used to solve DNDRP (DNDRP-LR).

In terms of future research, it would be interesting to apply this simultaneous methodology to more complex supply chains with more stages, considering the inventory at the production plants, or even considering the production process and raw materials replenishment. Furthermore it is possible to consider different levels of shared information between plants and warehouses, allowing to model the bullwhip effect and its impact on the distribution network design. Another unexplored extension of this methodology is the consideration of a more complex inventory model for a supply chain with multiple products and multiple periods. Finally, this approach can be incorporated into other strategic logistics and supply chain management problems, such as transportation network design, covering problems associated with facility location models.

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Appendix A

In this appendix we demonstrate the property used to solve sub-problems SP1 and SP2 (stated on Section 5.1). The Variables and parameters used in this appendix are the following:

U_i variance of demand served by warehouse i
CS_i safety stock cost for warehouse i
λ_i dual multipliers associated to every warehouse i
VT variance of the aggregated demand in the system

The property used in Section 5.1, states that searching the optimal solution for the SP1 consists in finding the warehouse t, that minimizes the value $CS_t\sqrt{VT} - \lambda_t VT$; if this minimum value is negative, then do the following:

$$U_i^* = \begin{cases} VT & \text{if } i = t \\ 0 & \text{if } i \neq t \end{cases} \tag{A.1}$$

The demonstration is based on mathematical induction and it is separated into two parts. Part I shows the property in the case of 2 warehouses. Part II shows the property in case of $N+1$ warehouses, assuming that the property is true for N warehouses. In this appendix we assume that CS_i; and λ_i, are non-negative parameters.

Part I

In this part of appendix we demonstrate the property for the smallest case, which considers two warehouses. In this case we must solve the following problem: [4]

$$\begin{aligned} \text{SP1}^2 \quad \text{Min} \quad & CS_1 \cdot \sqrt{U_1} - \lambda_1 \cdot U_1 + CS_2 \cdot \sqrt{U_2} - \lambda_2 \cdot U_2 \\ \text{s.t.:} \quad & \left.\begin{aligned} U_1 + U_2 = VT \\ U_1, U_2 \geqslant 0 \end{aligned}\right\} X \end{aligned} \tag{A.2}$$

If we characterize the feasible space, X, in term of its extreme points, given by

$$X^1 = \begin{pmatrix} U_1^1 \\ U_2^1 \end{pmatrix} = \begin{pmatrix} VT \\ 0 \end{pmatrix} \quad \text{and} \quad X^2 = \begin{pmatrix} U_1^2 \\ U_2^2 \end{pmatrix} = \begin{pmatrix} 0 \\ VT \end{pmatrix} \tag{A.3}$$

the feasible points, $X \in X$, can be represented by

$$X = \alpha X^1 + (1-\alpha)X^2 = \begin{pmatrix} \alpha VT \\ (1-\alpha)VT \end{pmatrix} \tag{A.4}$$

where $0 \leqslant \alpha \leqslant 1$. Consequently $ST1^2$ is equivalent to

$$\begin{aligned} \text{SP1}^2 \quad \text{Min} \quad & G(\alpha) = CS_1\sqrt{\alpha VT} - \lambda_1 \alpha VT + CS_2\sqrt{(1-\alpha)VT} - \lambda_2(1-\alpha)VT \\ \text{s.t.:} \quad & 0 \leqslant \alpha \leqslant 1 \end{aligned} \tag{A.5}$$

If we analyze $\frac{d^2G}{d\alpha^2}$, which can be written as follows:

$$\frac{d^2G}{d\alpha^2} = -\frac{CS_1 VT^2}{4\sqrt{\alpha VT}^3} - \frac{CS_2 VT^2}{4\sqrt{(1-\alpha)VT}^3} \tag{A.6}$$

it is negative for $0 < \alpha < 1$, and is undefined if $\alpha = 0$ or $\alpha = 1$. Thus it is concluded that $G(\alpha)$ is concave, and the optimal solution is always on the borders of the interval, which means that the optimal solution is $X = X^1$ or $X = X^2$, which finally is equivalent to

$$\begin{pmatrix} U_1^* \\ U_2^* \end{pmatrix} = \begin{pmatrix} VT \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} U_1^* \\ U_2^* \end{pmatrix} = \begin{pmatrix} 0 \\ VT \end{pmatrix} \tag{A.7}$$

Thus, property has been demonstrated for the smallest case, which considers 2 warehouses.

Part II

Here, we assume that the property is valid for the case of N warehouses (induction hypothesis). Then, the optimal solution of SP1^N, given by

$$\begin{aligned} \text{SP1}^N \quad \text{Min} \quad & \sum_{i=1}^{N} \left(CS_i\sqrt{U_i} - \lambda_i U_i \right) \\ \text{s.t.:} \quad & \sum_{i=1}^{N} U_i = VT \\ & U_1, U_2, \ldots, U_N \geqslant 0 \end{aligned} \tag{A.8}$$

[4] In this appendix, k in expression SP1^k, is the number of warehouses intervening in the sub-problem.

is found by doing:

$$U_i^* = \begin{cases} VT & i = w \\ 0 & i \neq w \end{cases} \tag{A.9}$$

for some $w \in \{1,\ldots,N\}$. We must demonstrate the property for the problem with $N+1$ warehouses:

$$\begin{aligned} \mathrm{SP1}^{N+1} \quad & \mathrm{Min} \quad \sum_{i=1}^{N+1} CS_i\sqrt{U_i} - \lambda_i U_i \\ & \text{s.t.:} \quad \sum_{i=1}^{N+1} U_i = VT \\ & \qquad\quad U_1,\ldots,U_N,U_{N+1} \geqslant 0 \end{aligned} \tag{A.10}$$

This problem can be decomposed as follows:

$$\begin{aligned} \mathrm{SP1}^{N+1} \quad & \mathrm{Min} \quad CS_{N+1}\sqrt{U_{N+1}} - \lambda_{N+1}U_{N+1} + V(U_{N+1}) \\ & \text{s.t.:} \quad 0 \leqslant U_{N+1} \leqslant VT \end{aligned} \tag{A.11}$$

where

$$\begin{aligned} V(U_{N+1}) = \quad & \mathrm{Min} \quad \sum_{i=1}^{N} CS_i\sqrt{U_i} - \lambda_i U_i \\ & \text{s.t.:} \quad \sum_{i=1}^{N} U_i = VT - U_{N+1} \\ & \qquad\quad U_1,U_2,\ldots,U_N \geqslant 0 \end{aligned} \tag{A.12}$$

Solving $V(U_{N+1})$ is equivalent to solve $\mathrm{SP1}^N$ replacing VT by $VT - U_{N+1}$. Then, considering the induction hypothesis, the optimal solution of $V(U_{N+1})$ is given by

$$U_i^* = \begin{cases} VT - U_{N+1} & i = w \\ 0 & i \neq w \end{cases} \tag{A.13}$$

for some $w \in \{1,\ldots,N\}$. Thus, $\mathrm{SP1}^{N+1}$ can be written as follows:

$$\begin{aligned} \mathrm{SP1}^{N+1} \quad & \mathrm{Min} \quad CS_{N+1}\sqrt{U_{N+1}} - \lambda_{N+1}U_{N+1} + CS_w\sqrt{U_w} - \lambda_w U_w \\ & \text{s.t.:} \quad U_{N+1} + U_w = VT \\ & \qquad\quad U_{N+1},U_w \geqslant 0 \end{aligned} \tag{A.14}$$

This problem has the same structure of $\mathrm{SP1}^2$, then optimal solution for $\mathrm{SP1}^{N+1}$, based on Part I, can be obtained choosing the warehouse t, between $N+1$ and w, so that if $CS_{N+1}\sqrt{VT} - \lambda_{N+1}VT \leqslant CS_w\sqrt{VT} - \lambda_w VT$, then $t = N+1$, and otherwise $t = w$. We have then demonstrated the property used for solving sup-problem SP1.

Appendix B

In this appendix we show the generic version of the LR heuristic used to solve the DNDRP model. LR heuristic was implemented in Visual C++ 5.0, considering an interface with LINGO 6.0, to solve the sub-problem SP3.

begin
  $\lambda_i := 0$ and $\mu_i := 0$, for every $i = 1, \ldots, N$;
  $k := 0$;
  $l := 0$;
  $S1 := \varepsilon 1 + 1$;
  $S2 := \varepsilon 2 + 1$;
  while $k \leqslant N\max 1$ and $l \leqslant N\max 2$ and $S1 > \varepsilon 1$ and $S2 > \varepsilon 2$ **do**
  begin
    solve the sub-problems $\mathrm{SP1}^k$ and $\mathrm{SP2}^k$ (based on the procedures stated in Section 5.1);
    compute violations for every relaxed constraints, VU^k and VD^k, based on Eq. (33);
    compute lower and upper bound for every iteration k, $Z_k^{\mathrm{Inf}} y Z_k^{\mathrm{Sup}}$, respectively, using Eqs. (37) and (39);
    update Lagrangian multipliers, λ_i, and μ_i for $i = 1, \ldots, N$, according to the expression (5.16);
    update counters k and l, and recalculate $S1$ and $S2$.
  end
end

k is a counter for the algorithm's iterations, which has a maximum value of $N\max 1$. l counts the consecutive iterations in which the upper bound has not been improved, for which there is a limit given for $N\max 2$. S1 computes the relative difference between lower and upper bounds for every iteration, for which is required a maximum value given by $\varepsilon 1$. $S2$ computes the maximum relative difference between two consecutive values of λ_i^k and μ_i^k, for $i = 1, \ldots, N$. This maximum difference has a threshold given by $\varepsilon 2$ (the heuristic stops when this value is reached). The first two convergence criteria are established when the problem has a *duality gap* [5] greater than $\varepsilon 1$.

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[5] Duality gap is the difference between optimal value of dual and primal problem, which can be positive for integer and mixed problems.

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